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THE SCATTERING MATRIX IN A WAVEGUIDE WITH ELASTIC WALLS*

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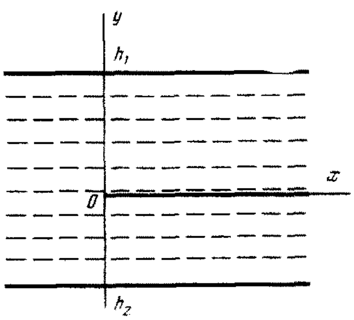
The spectrum of normal waves is studied and the scattering matrix is determined for a planar waveguide with elastic walls and with an elastic semi-infinite plate situated within it. The mechanical mode of behaviour of elastic plates is described using the general-type differential operators. Problems of this type belong to the class of the boundary contact problems /1, 2/. The unique solvability of these problems requires the formulation of additional boundary contact conditions describing the mechanical behaviour of the edge of the semi-infinite plate situated within the waveguide. The regularization of the integrals appearing when the general-type boundary contact conditions are satisfied is indicated.

1. Formulation of the problem. We seek a solution of the following two-dimensional homogeneous Helmholtz equation:

$$\partial^2 P/\partial x^2 + \partial^2 P/\partial y^2 + k^2 P = 0 \tag{1.1}$$

in the strip $-\infty < x < +\infty$, $h_2 < y < h_1$ with a cut $y = 0, x > 0$ (see the figure), describing the distribution of the pressure $P(x, y)$ when the system is excited by a given acoustic field $P_0(x, y)$. Here $k = \omega/c$ is the wave number, ω is the angular frequency; here and henceforth the dependence of the wave processes on time, chosen here in the form $\exp(-i\omega t)$, is neglected; c is the velocity of sound in the medium.



The mechanical behaviour of the walls of the waveguide, i.e. of elastic plates, is described by the following boundary conditions:

$$L_j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) P(x, h_j) = 0, \quad -\infty < x < +\infty, \quad j = 1, 2 \tag{1.2}$$

$$\left(L_j\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = (-1)^j M_{1j}\left(-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial y} + M_{2j}\left(-\frac{\partial^2}{\partial x^2}\right)\right)$$

A thin elastic plate is situated on the ray $y = 0, x > 0$ and it executes antisymmetric oscillations described by the boundary conditions $(x > 0)$

$$\frac{\partial P(x, +0)}{\partial y} = \frac{\partial P(x, -0)}{\partial y} \tag{1.3}$$

$$M_{13}\left(-\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial y} P(x, 0) + M_{23}\left(-\frac{\partial^2}{\partial x^2}\right)[P(x, +0) - P(x, 0)] = 0 \tag{1.4}$$

Condition (1.3) describes the equality of the displacements of the upper (lower) surface of the plate $u(x) = (\rho_0\omega^2)^{-1}\,\partial P(x, \pm 0)/\partial y$, ρ_0 is the fluid density. We note that condition (1.3) holds on the continuation of the plate median $y = 0, x < 0$, as well as the condition thatt the pressure is continuous

$$P(x, +0) = P(x, -0), \quad x < 0 \tag{1.5}$$

Here $M_{1j}(-\partial^2/\partial x^2)$, $M_{2j}(-\partial^2/\partial x^2)$ $(j = 1, 2, 3)$ are polynomials whose coefficients depend on the mechanical properties of the elastic materials of which the waveguide walls are made.

We illustrate all this by describing the form of the differential operators for different types of the waveguide walls: $M_{1j} = 1$, $M_{2j} = 0$ (perfectly rigid walls); $M_{1j} = 0$, $M_{2j} = 1$ (perfectly pliable walls); $M_{1j} = \partial^2/\partial x^2 + K_j^2$, $M_{2j} = \rho_0\omega^2/N_j$ (elastic membranes); $M_{1j} = \partial^4/\partial x^4 - \varkappa_j^4$, $M_{2j} = \rho_0\omega^2/D_j$ (elastic, flexurally oscillating plates). Here K_j is the wave number of the waves within the membrane, $K_j = \rho_j/N_j$, ρ_j is the linear density of the membrane (plate), N_j is the tensile force in the membrane, $\varkappa_j$ is the wave number of the flexural waves in the plate situated in vacuo, and $\varkappa_j = \rho_j\omega^2/D_j$, D_j is the flexural rigidity of the plate.

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We note that in the case of elastic plates the order m_{1j} of the polynomials M_{1j} is greater than the order m_{2j} of the polynomials M_{2j}.

The solution $P(x, y)$ sought satisfies the principle or limit absorption (the case of real k is regarded as a passage to the limit as $\operatorname{Im} k \to 0$), continuously in the region in question up to the boundaries, and satisfies the Meixner condition "on the edge".

The problem has a unique solution for the Dirichlet conditions (perfectly pliable walls), for the Neumann conditions (perfectly rigid walls) and for the conditions of third kind on the ray $y = 0, x > 0$. If at least one of the numbers m_{1j} and m_{2j} is different from zero, then the solution which we call the general one /1/, loses its uniqueness and contains a number of arbitrary constants. The values of these constants are obtained from the requirement that additional boundary contact conditions /2, 3/ must be satisfied. The conditions specify the mechanical mode of behaviour of the edge of the semi-infinite plate.

For example, an elastic membrane whose edge is clamped, the boundary contact condition has the form $\partial P(+0, 0)/\partial y = 0$. In the case of a flexurally oscillating elastic plate whose edge is clamped, the conditions that the displacement and the angle of rotation of the plate should be zero, $\partial P(+0, 0)/\partial y = 0$ and $\partial^2 P(+0, 0)/\partial y \partial x = 0$, both hold. If the edge of the plate is free, then the boundary contact conditions specify the fact that the bending moment and the shear force $\partial^3 P(+0, 0)/\partial y \partial x^2 = 0$ and $\partial^4 P(+0, 0)/\partial y \partial x^3 = 0$ are both zero.

2. Spectrum of normal waves in the waveguide. We shall seek the normal waves $Q(x, y)$ of the right waveguide $(x > 0)$ in the form $Q(x, y) = A_j G_j(\lambda^2, y) e^{i\lambda x}$. Here and henceforth we have for all functions depending on y, $j = 1$, if $0 < y < h_1$, $j = 2$, if $h_2 < y < 0$. The function

$$G_j(\lambda^2, y) = M_{1j}(\lambda^2) \operatorname{ch} \gamma (y - h_j) - (-1)^j M_{2j}(\lambda^2) \gamma^{-1} \operatorname{sh} \gamma (y - h_j),$$
$$\gamma = (\lambda^2 - k^2)^{1/2}$$

is chosen in such a manner, that conditions (1.1) and (1.2) are satisfied for $Q(x, y)$.

The boundary conditions (1.3), (1.4) lead to a system of linear algebraic equations for the constants

$$A_1 L_1(\lambda^2) - A_2 L_2(\lambda^2) = 0 \tag{2.1}$$
$$A_1 [M_{13}(\lambda^2) L_1(\lambda^2) + M_{23}(\lambda^2) T_1(\lambda^2)] - A_2 M_{23}(\lambda^2) T_2(\lambda^2) = 0$$
$$(L_j(\lambda^2) = \partial G_j(\lambda^2, 0)/\partial y, \quad T_j(\lambda^2) = G_j(\lambda^2, 0))$$

The condition for a non-zero solution of system (2.1) to exist yields a dispersion equation for determining the spectrum of wave numbers of the normal waves of the right waveguide

$$\Delta_1(\lambda^2) = M_{13}(\lambda^2) L_1(\lambda^2) L_2(\lambda^2) + M_{23}(\lambda^2) \Delta_2(\lambda^2) = 0 \tag{2.2}$$
$$(\Delta_2(\lambda^2) = T_1(\lambda^2) L_2(\lambda^2) - T_2(\lambda^2) L_1(\lambda^2))$$

The function $\Delta_1(\lambda^2)$ is even; therefore we can limit ourselves, when studying the roots of (2.2) (wave numbers of the normal waves of the right waveguide), to those roots which lie in the upper half-plane of the complex variable λ when $\operatorname{Im} k > 0$.

For the left waveguide $(x < 0)$, not containing a middle plate, the dispersion equation for the wave numbers of the normal waves can be obtained from (2.2), provided that we write formally $M_{23}(\lambda^2) = 1$, $M_{13}(\lambda^2) = 0$. The corresponding dispersion equation has the form

$$\Delta_2(\lambda^2) = 0 \tag{2.3}$$

and its root separate into two groups. The roots in the first group β_{1l} form a denumerable set and approach asymptotically, as l increases, the roots of the dispersion equation for a waveguide of width $h_1 - h_2$, with rigid walls

$$\beta_{1l} \sim i \left[\left(\frac{\pi l}{h_1 - h_2} \right)^2 - k^2 \right]^{1/2}, \quad l \to \infty$$

The process can be observed if we transform Eq.(2.3) to the form

$$\gamma \operatorname{th} \gamma (h_1 - h_2) = \gamma^2 [M_{11}(\lambda^2) M_{22}(\lambda^2) + M_{21}(\lambda^2) M_{12}(\lambda^2)] \times \tag{2.4}$$
$$[\gamma^2 M_{11}(\lambda^2) M_{12}(\lambda^2) + M_{21}(\lambda^2) M_{22}(\lambda^2)]^{-1}$$

The right side of Eq.(2.4) tends to zero as $|\lambda| \to \infty$; therefore for sufficiently large $|\lambda|$ we arrive at the dispersion equation for a waveguide of width $h_1 - h_2$ with rigid walls $\gamma \operatorname{th} \gamma (h_1 - h_2) = 0$. We shall call the roots belonging to this group the waveguide roots. The roots belonging to the second group can be conveniently studied under the assumption that the density of the acoustic medium is low. The root approach the roots of the polynomials $M_{11}(\lambda^2)$, $M_{12}(\lambda^2)$ which represent the wave numbers of theplates in vacuo. We shall call them the roots belonging to zero group and denote them by β_{0l}, $l = 1, 2, \ldots, m_{11} + m_{12}$.

The roots of the dispersion equation of the right waveguide (2.2) are dealt with in exactly the same manner. When the density of the acoustic medium decreases, the zero group of roots α_{0l} $(l = 1, 2, \ldots, m_{11} + m_{12} + m_{13})$ approaches the set of roots of the polynomials $M_{11}(\lambda^2)$, $M_{12}(\lambda^2)$, $M_{13}(\lambda^2)$. The waveguide group of roots splits into two subgroups $(l \to \infty)$

$$\alpha_{1l} \sim i \left[\left(\frac{\pi l}{h_1} \right)^2 - k^2 \right]^{1/2}, \quad \alpha_{2l} \sim i \left[\left(\frac{\pi l}{h_2} \right)^2 - k^2 \right]^{1/2}$$

Henceforth, we shall assume for convenience that $\lambda_{1l}\ \lambda_{2l}\ (l = 1, 2, \ldots)$ denote the set of roots of the functions $\Delta_1(\lambda^2), \Delta_2(\lambda^2)$ beginning, respectively, with real numbers (when $\operatorname{Im} k = 0$).

3. General solution of the problem. We will consider the diffraction of normal waves of the right (left) waveguide by a semi-infinite plate within the waveguide. We write the pressure field in the form

$$P(x, y) = P_0(x, y) + P_1(x, y)$$

where

$$P_0(x, y) = (E(\lambda_{1n}))^{-1/2} G_0(\lambda_{1n}^2, y) \exp(-i\lambda_{1n} x)$$
$$G_0(\lambda^2, y) = G_j(\lambda^2, y)/L_j(\lambda^2))$$

is the normal wave travelling along the right waveguide from the direction of positive x, with wave number n, $P_1(x, y)$ is the diffraction field generated by it. The factor $(E(\lambda_{1n}))^{-1/2}$ is chosen so that the energy flux aveaged over a period, transported by the propagating normal wave with number n ($\operatorname{Im}\lambda_{1n} = 0$ when $\operatorname{Im} k = 0$) is equal to one. The flux represents the sum of the energy fluxes averaged over a period, transported by this wave along the acoustic medium Π_0, i.e.

$$\Pi_0 = \frac{1}{2\rho_0\omega} \operatorname{Im} \int_{h_2}^{h_1} P_0(x, y) \frac{\partial P_0^*(x, y)}{\partial x}\, dy$$

and the energy fluxes transported along the elastic plates. When the plate is oscillating flexurally, the fluxes are computed from the formula

$$\Pi = \sum_{j=0}^{2} \frac{D_j}{2\rho_0^2\omega^3} \left(\frac{\partial P_0(x, h_j)}{\partial y} \frac{\partial^4 P_0^*(x, h_j)}{\partial y\, \partial x^3} - \frac{\partial^2 P_0(x, h_j)}{\partial y\, \partial x} \frac{\partial^3 P_0^*(x, h_j)}{\partial y\, \partial x^2} \right)$$

where $h_0 = 0$.

We shall seek the diffraction field in the form of a Fourier integral

$$P_1(x, y) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} p(\lambda)\, G_0(\lambda^2, y)\, e^{i\lambda x}\, d\lambda \tag{3.1}$$

for which the conditions (1.1)−(1.3) hold automatically.

Satisfy the conditions (1.4) and (1.5) we arrive, using standard methods, at the inhomogeneous Riemann problem for analyic functions. Using known procedures /4, 5/ to solve it, we obtain the general solution of the problem

$$p(\lambda) = \frac{L_1(\lambda^2) L_2(\lambda^2) \varphi^-(\lambda)}{\Delta_1(\lambda^2)} \left(f_n(\lambda) - \frac{(E(\lambda_{1n}))^{-1/2} \varphi^+(\lambda_{1n}) \Delta_2(\lambda_{1n}^2)}{(\lambda + \lambda_{1n}) L_1(\lambda_{1n}^2) L_2(\lambda_{1n}^2)} \right)$$

where $f_n(\lambda)$ is an $m_{13} - 1$-th degree polynomial; the coefficients of the polynomial depend on λ_{1n}, and will be determined in Sect.4 and $\varphi^\pm(\pm\lambda)$ is the result of factorizing the function $\varphi(\lambda) = \Delta_1(\lambda^2)/\Delta_2(\lambda^2)$. The functions $\varphi^-(\lambda)$ and $\varphi^+(-\lambda)$ are analytic in the upper and lower half-plane of the complex variable λ respectively

$$\varphi^-(\lambda) = (\varphi(0))^{1/2} \exp\left(\frac{i\lambda}{\pi}\left(h_1 \ln \frac{h_1 - h_2}{h_1} - h_2 \ln \frac{h_2 - h_1}{h_2}\right)\right) \times$$
$$\prod_{l=1}^{m_1} q(\lambda, \alpha_{0l}) \prod_{l=1}^{\infty} q(\lambda, \alpha_{1l}) \exp\left(i \frac{\lambda h_1}{l\pi}\right) \prod_{l=1}^{\infty} q(\lambda, \alpha_{2l}) \exp\left(-i \frac{\lambda h_2}{l\pi}\right) \times$$
$$\left[\prod_{l=1}^{m_2} q(\lambda, \beta_{0l}) \prod_{l=1}^{\infty} q(\lambda, \beta_{1l}) \exp\left(i \frac{\lambda(h_1 - h_2)}{l\pi}\right)\right]^{-1}$$
$$(m_1 = m_{11} - m_{12} + m_{13},\ m_2 = m_{11} + m_{12},\ q(\lambda, \mu) = 1 + \lambda/\mu)$$

and we have $\varphi^\pm(\lambda) = O(\lambda^{m_{13} - 1/2})$, $|\lambda| \to \infty$ in the upper half-plane.

The problem of the excitation of the diffraction field by a normal wave arriving along the left waveguide is solved in exactly the same manner.

4. Boundary contact conditions. To find the coefficients m_{13} of the polynomial $f_n(\lambda)$ which are, so far, arbitrary, we must specify m_{13} boundary contact conditions determining the mechanical mode of behaviour of the semi-infinite plate edge. The general form of these conditions is

$$R_l P(+0, 0) = \lim_{x \to +0} \left\{ F_{1l}\left(-i \frac{\partial}{\partial x}\right) \frac{\partial P(x, 0)}{\partial y} + F_{2l}\left(-i \frac{\partial}{\partial x}\right) [P(x, +0) - P(x, -0)] \right\}, \quad l = 1, 2, \ldots, m_{13} \tag{4.1}$$

Here $F_{1l}(-i\partial/\partial x)$ and $F_{2l}(-i\partial/\partial x)$ are polynomials whose coefficients are determined by the mechanical properties of the system.

Writing the explicit expansion of the polynomial $f_n(\lambda)$ in powers of λ

$$f_n(\lambda) = a_0 + a_1\lambda + \ldots + a_{m_{13}-1}\lambda^{m_{13}-1}$$

and imposing the boundary contact conditions (4.1) on the field $P(x, y)$ we obtain, after a series of transformations, the following system of linear algebraic equations for determining the boundary contact constants a_m $(m = 0, 1, \ldots, m_{13} - 1)$:

$$\sum_{m=0}^{m_{13}-1} a_m H_{lm} = I_{ln}, \quad l = 1, 2, \ldots, m_{13}$$

where

$$H_{lm} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\varphi^+(\lambda)}{\Delta_1(\lambda^2)} \lambda^m (F_{1l}(\lambda) L_1(\lambda^2) L_2(\lambda^2) + F_{2l}(\lambda) \Delta_2(\lambda^2)) e^{+i0\lambda} d\lambda \tag{4.2}$$

$$I_{ln} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\varphi^+(\lambda_{1n}) M_{13}(\lambda_{1n}^2) \varphi^+(\lambda)}{M_{23}(\lambda_{1n}^2)(\lambda + \lambda_{1n}) \Delta_1(\lambda^2)} (F_{1l}(\lambda) L_1(\lambda^2) L_2(\lambda^2) + \tag{4.3}$$

$$F_{2l}(\lambda) \Delta_2(\lambda^2)) e^{+i0\lambda} d\lambda - F_{1l}(-\lambda_{1n}) + F_{2l}(-\lambda_{1n}) \frac{M_{13}(\lambda_{1n}^2)}{M_{23}(\lambda_{1n}^2)}$$

$$\left(\int_{-\infty}^{+\infty} \xi(\lambda) e^{+i0\lambda} d\lambda = \lim_{x \to +0} \int_{-\infty}^{+\infty} \xi(\lambda) e^{i\lambda x} d\lambda \right)$$

A formal passage to the limit in (4.2) and (4.3) leads to divergent integrals. To regularize the integrals in (4.2) and (4.3) we will require that the following relation holds (its meaning was discussed in /2/):

$$F_{1l}(\lambda) M_{23}(\lambda^2) - F_{2l}(\lambda) M_{13}(\lambda^2) = o(\lambda^{2m_{13}}) \tag{4.4}$$
$$|\lambda| \to \infty \ (l = 1, 2, \ldots, m_{13})$$

specifying the relation connecting the boundary and boundary contact operators. Then

$$H_{lm} = \sum_{\mathrm{Im}\,\lambda > 0} \mathrm{Res}\left[\lambda^m \frac{\varphi^+(\lambda) F_{1l}(\lambda)}{M_{13}(\lambda^2)}\right] - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \lambda^m \frac{\varphi^+(\lambda) \Delta_2(\lambda^2)}{M_{13}(\lambda^2) \Delta_1(\lambda^2)} (F_{1l}(\lambda) M_{23}(\lambda^2) - F_{2l}(\lambda) M_{13}(\lambda^2)) d\lambda$$

We regularize I_{ln} in exactly the same manner.

5. The scattering matrix.

The scattered field generated by the normal wave arriving along the left waveguide is sought in the form of the Fourier integral (3.1) using the same methods as before.

Expanding the integrals in (3.1) and in the analogous integral for the case when the field within the waveguide is generated by the normal wave of the left waveguide, in the sums of residues, we construct the transmission matrix S^{21} (S^{12}) and the reflection matrix S^{22} (S^{11}) of the normal waves generated in the left (and respectively the right) waveguide. The dispersion matrix

$$S = \begin{Vmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{Vmatrix}$$

is composed of blocks with elements

$$S_{mn}^{11} = \left(f_n(-\lambda_{2m}) + \frac{\tau_{1n}}{-\lambda_{2m} + \lambda_{1n}} \right) \nu_{2m} \Psi(\lambda_{2m}, \lambda_{1n})$$

$$S_{mn}^{12} = \left(f_n(\lambda_{1m}) + \frac{\tau_{1n}}{\lambda_{1m} + \lambda_{1n}} \right) \nu_{1m} \Psi(\lambda_{1m}, \lambda_{1n})$$

$$S_{mn}^{21} = \left(g_n(-\lambda_{2m}) + \frac{\tau_{2n}}{\lambda_{2m} + \lambda_{1n}} \right) \nu_{2m} \Psi(\lambda_{2m}, \lambda_{2n})$$

$$S_{mn}^{22} = \left(g_n(\lambda_{1m}) + \frac{\tau_{2n}}{-\lambda_{1m} + \lambda_{2n}} \right) \nu_{1m} \Psi(\lambda_{1m}, \lambda_{2n})$$

where

$$\tau_{1n} = \varphi^+(\lambda_{1n}) M_{13}(\lambda_{1n}^2) / M_{23}(\lambda_{1n}^2), \quad \tau_{2n} = M_{13}(\lambda_{2n}^2) / \varphi^+(\lambda_{2n})$$

$$\nu_{1m} = L_1(\lambda_{1m}^2) L_2(\lambda_{1m}^2) \varphi^+(\lambda_{1m}) \left[\frac{d}{d\lambda} \Delta_1(\lambda^2) \Big|_{\lambda = \lambda_{1m}} \right]^{-1}$$

$$\nu_{2m} = L_1(\lambda_{2m}^2) L_2(\lambda_{2m}^2) \left[\varphi^+(\lambda_{2m}) \frac{d}{d\lambda} \Delta_2(\lambda^2) \Big|_{\lambda = \lambda_{2m}} \right]^{-1}$$

$$\Psi(\lambda, \mu) = [E(\lambda^2)/E(\mu^2)]^{1/2}$$

The coefficients of the polynomial $g_n(\lambda) = b_0 + b_1\lambda + \ldots + b_{m_{13}-1}\lambda^{m_{13}-1}$ should be found from the system of algebraic equations

$$\sum_{m=0}^{m_{13}-1} b_m H_{lm} = J_{ln}, \quad l = 1, 2, \ldots, m_{13}$$

$$J_{ln} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\tau_{2n} \varphi^+(\lambda)}{(\lambda - \lambda_{2n}) \Delta_1(\lambda^2)} (F_{1l}(\lambda) L_1(\lambda^2) L_2(\lambda^2) + F_{2l}(\lambda) \Delta_2(\lambda^2)) e^{+i0\lambda} d\lambda$$

The contour Γ circumvents from below all poles of the integrand lying, when $\operatorname{Im} k > 0$, in the upper half-plane of the complex variable λ except for $\lambda = \lambda_{2n}$.

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